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THE COMPETITION-COMMON ENEMY GRAPH OF A DIGRAPH

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The competition graph of a digraph was first defined in 1968 by Cohen in the study of ecosystems. The competition graph essentially relates any two species which have a common prey. In this paper, a competition-common enemy graph of a digraph is defined and studied. As the term suggests, it relates any two species which have a common prey and a common enemy. Results analogous to those found for competition graphs are obtained.

Introduction

In 1968 Cohen [1] introduced competition graphs associated with food web models of ecosystems. The *competition graph* of a digraph $D=(V,A)$ is the graph $G=(V,E)$ where $xy\in E$ if and only if $x\neq y\in V$ and $\overrightarrow{xz}, \overrightarrow{yz}\in A$ for some $z\in V$. Now define the *competition-common enemy graph* (*CCE graph*) of a digraph $D=(V,A)$ as a graph $G=(V,E)$ where $xy\in E$ if and only if $x\neq y\in V$ and $\overrightarrow{wx}, \overrightarrow{wy}, \overrightarrow{xz}, \overrightarrow{yz}\in A$ for some $w,z\in V$.

The first part of this paper defines the double competition number of a graph G ($\mathrm{dk}(G)$) and characterizes graphs with various double competition numbers. Surprisingly, no graphs with $\mathrm{dk}(G)>2$ are found. Much of the work parallels that done by Roberts [7] on the competition number of a graph G, ($\mathrm{k}(G)$).

The second section dwells specifically on the double competition number for *transitive*, acyclic digraphs in an attempt to discover a graph with double competition number greater than two.

Lastly, graphs which are competition-common enemy graphs of acyclic digraphs, of digraphs without loops, of digraphs with loops allowed, and of digraphs with loops on all vertices are characterized.

The reader should note that all sets considered are finite and that a clique is a *maximal* complete subgraph.

* The material in this paper is taken from Chaper 1 of my Ph.D. dissertation.

1. The double competition number

In 1978 Roberts [7] observed that starting with any graph G, a competition graph is obtained by adding sufficiently many isolated vertices to G. Following this observation, it was natural for him to define $\mathrm{k}(G)$, the competition number of G, to be the smallest integer k such that $G \cup I_k$ is a competition graph of an acyclic digraph, where I_k is a set of isolated vertices added to G. Note that the competition number of a graph is defined for graphs which are competition graphs of acyclic digraphs. Although competition graphs arising from other types of digraphs were later studied, the initial restriction to acyclic digraphs arose from the basic assumption that food web models of ecosystems are acyclic.

Analogously define $\mathrm{dk}(G)$, the double competition number of G, to be the smallest integer k such that $G \cup I_k$ is a CCE graph of an acyclic digraph, where I_k is again a set of k isolated vertices added to G. Note that $\mathrm{dk}(G)$ is well-defined since, given any graph G, a CCE graph arising from an acyclic digraph can be constructed as follows.

For each edge $\alpha = xy$ in G, add a pair of isolated vertices $\{x_\alpha, y_\alpha\}$ to G. Then define the digraph D such that

$$V(D) = V(G) \cup \{x_\alpha, y_\alpha : \alpha \in E(G)\} \quad \text{and}$$

$$E(D) = \{\overrightarrow{x_\alpha x}, \overrightarrow{x_\alpha y}, \overrightarrow{x y_\alpha}, \overrightarrow{y y_\alpha} : \alpha = xy \in E(G)\}.$$

The above construction gives an upper bound for $\mathrm{dk}(G)$, namely $2 \cdot |E(G)|$. But this easily can be improved. Note that for each edge $\alpha = xy$ in G, there must be a common prey and a common enemy in D. First find $\mathrm{k}(G) = k$, which adds k isolated vertices to G. If D is the digraph with competition graph $G \cup I_k$, then endpoints of each edge in G have a common prey in D. A common enemy results by adding one more isolated vertex to G with arcs in D from it to each original vertex of G. Hence,

$$\mathrm{dk}(G) \le \mathrm{k}(G) + 1.$$

For any graph without isolated vertices a lower bound for $\mathrm{dk}(G)$ is immediate. As Harary, Norman, and Cartwright proved in [5], if D is any acyclic digraph and $|V(D)| = n$, then integers $1, 2, \ldots, n$ can be assigned to the vertices of D such that every arc goes from a lower number of a higher one. Thus v_1 has no incoming arcs and v_n has no outgoing arcs. The CCE graph G of D clearly has at least two isolated vertices, namely v_1 and v_n, and thus for graphs without isolated vertices $\mathrm{dk}(G) \ge 2$.

The above observations lead to the following proposition.

Proposition 1. *For $G = K_n$, $n \ge 2$, $\mathrm{dk}(G) = 2$.*

Proof. From the above observations and the fact that $\mathrm{k}(K_n) = 1$ we have $2 \le \mathrm{dk}(K_n) \le \mathrm{k}(K_n) + 1 = 2$. $\square$

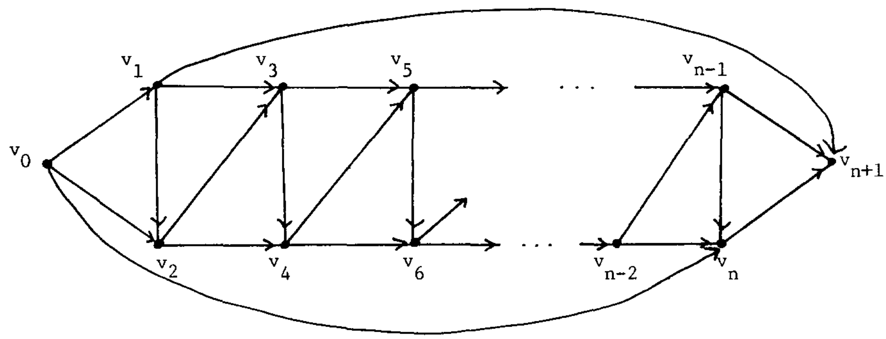

Fig. 1.

Roberts [7] proved that if $G=C_n$, $n>3$, then $\mathrm{k}(G)=2$. The next theorem shows that $\mathrm{dk}(C_n)=2$ for $n\geq 3$. Thus, C_4 is an example where $\mathrm{dk}(G)<\mathrm{k}(G)+1$. In general, if $G=C_n$, $n\geq 4$, then $\mathrm{dk}(G)=\mathrm{k}(G)$. This follows directly from the next theorem. However, $\mathrm{k}(C_3)=1$ while $\mathrm{dk}(C_3)=2$.

Theorem 1. *For* $G=C_n$, $n\geq 3$, $\mathrm{dk}(G)=2$.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G such that $E(G)=\{v_iv_{i+1}, v_nv_1 : i=1, 2, \ldots, n-1\}$. Since G has no isolated vertices, $\mathrm{dk}(G)\geq 2$. Add isolated vertices v_0 and v_{n+1} to G. Define D to be the digraph with $V(D)=V(G)\cup\{v_0, v_{n+1}\}$ and

$$E(D)=\{\overrightarrow{v_0v_n}, \overrightarrow{v_1v_{n+1}}\}\cup\{\overrightarrow{v_iv_{i+1}} : i=0,1,\ldots,n\}\cup\{\overrightarrow{v_iv_{i+2}} : i=0,1,\ldots,n-1\}.$$

D is shown in Fig. 1. The reader can check that D is acyclic and has CCE graph $G\cup I_2$, where $I_2=\{v_0, v_{n+1}\}$. $\square$

The following result follows by a similar argument and an appropriate digraph is given in Fig. 2.

Theorem 2. *For* $G=P_n$, $n\geq 2$, $\mathrm{dk}(G)=2$.

We have already observed that $\mathrm{dk}(G)\leq \mathrm{k}(G)+1$. If G is a competition graph, then $\mathrm{k}(G)=0$ so $\mathrm{dk}(G)\leq 1$. Example 1 below shows that it may actually be the case that $\mathrm{dk}(G)=0$ if G is a competition graph with more than one isolated vertex.

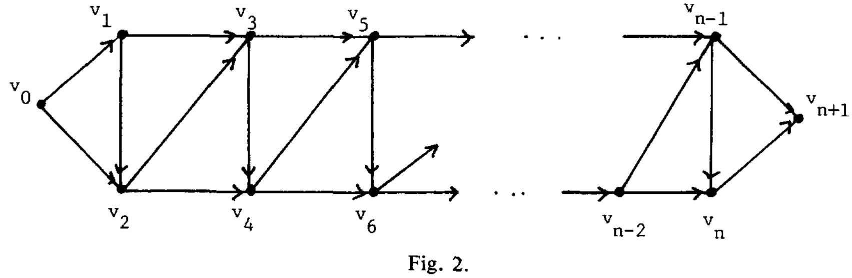

Fig. 2.

Example 1. Consider the graph $G=C_4 \cup I_2$. It follows from Roberts [7] that G is a competition graph. Moreover, by Theorem 1, $\mathrm{dk}(C_4)=2$. Hence, $\mathrm{dk}(G)=0$.

The following corollary is immediate from the above remarks.

Corollary 1. $\mathrm{dk}(G)=\mathrm{k}(G)+1$ *if one of the following holds:*
(i) $\mathrm{k}(G)=0$ *and* G *has exactly one isolated vertex.*
(ii) $\mathrm{k}(G)=1$ *and* G *has no isolated vertices.*

The following Theorem of Roberts [7] is used to obtain an example where $\mathrm{dk}(G)<\mathrm{k}(G)$.

Theorem 3. *If* G *is connected,* $|V(G)|>1$, *and* G *has no triangles, then* $\mathrm{k}(G)=|E(G)|-|V(G)|+2$.

Note that it follows from Theorem 3 and Theorem 2 that if $G=P_n$, $n \geq 2$, then $\mathrm{dk}(G)=\mathrm{k}(G)+1$.

Example 2. Let G be the graph in Fig. 3(a). G satisfies the conditions of Theorem 3 and thus $\mathrm{k}(G)=6-5+2=3$. However, the digraph shown in Fig. 3(b) has CCE graph $G \cup I_2$, where $I_2=\{v_0, v_6\}$. Thus, $\mathrm{dk}(G)<\mathrm{k}(G)$.

Corollaries 3 and 4 of Roberts [7] prove that every rigid circuit (chordal) graph G has $\mathrm{k}(G) \leq 1$ and every interval graph G has $\mathrm{k}(G) \leq 1$. In the last corollary of the same paper, Roberts shows that for a triangle-free, connected graph G with $|V|>1$, $\mathrm{k}(G)=1$ if and only if G is a tree.

From these corollaries and our previous observations, we have

Corollary 2. *Every chordal graph* G *has* $\mathrm{dk}(G) \leq 2$.

Corollary 3. *Every interval graph* G *has* $\mathrm{dk}(G) \leq 2$.

Corollary 4. *If* G *is a nontrivial tree, then* $\mathrm{dk}(G)=2$.

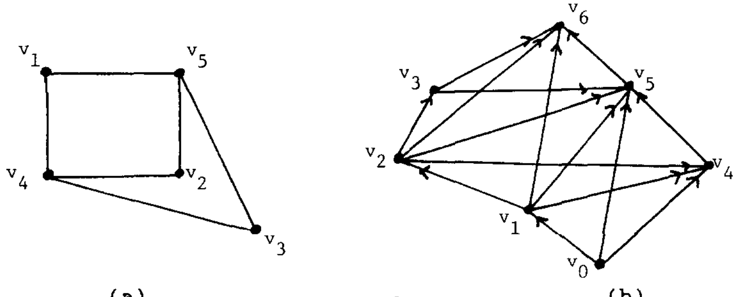

Fig. 3.

I have not yet found a graph for which $\mathrm{dk}(G)>2$. In searching for such a graph G, I have looked mainly at connected graphs without triangles. For these graphs, the largest complete subgraph is an edge and hence, each edge needs a distinct pair associated with it as a common enemy and common prey in D. Moreover, for these graphs $\mathrm{k}(G)=|E(G)|-|V(G)|+2$. In Chapter 1 of my Ph.D. dissertation [2] I investigated various triangle-free graphs G with $\mathrm{k}(G)=3,4,5,6$ and 7 and found that $\mathrm{dk}(G)=2$ for each of them. The following theorem is a result of this.

Theorem 4. *If $G=K_{m,n}$, the complete bipartite graph with $|V|=m+n$, then* $\mathrm{dk}(G)=2$.

Proof. Let $v_1, v_2, \ldots, v_m$ be the vertices in one partite set of G and $v_{m+1}, v_{m+2}, \ldots, v_{m+n}$ be the vertices in the other. Define a digraph D with $V(D)=V(G)\cup\{v_0, v_{m+n+1}\}$ and

$$E(D)=\{\overrightarrow{v_0v_j}: m+1\leq j\leq m+n\}\cup\{\overrightarrow{v_iv_j}: 1\leq i\leq m, m+1\leq j\leq m+n+1\}$$
$$\cup\{\overrightarrow{v_iv_{i+1}}: 0\leq i\leq m-1, m+1\leq i\leq m+n\}.$$

Then it is easy to check that D has CCE graph $G\cup\{v_0, v_{m+n+1}\}$ and so $\mathrm{dk}(G)=2$. $\square$

2. Competition number and double competition number for transitive acyclic digraphs

Define a *transitive acyclic digraph* $D=(V,A)$ to be an acyclic digraph such that $\overrightarrow{v_iv_j}, \overrightarrow{v_jv_k}\in A$ implies $\overrightarrow{v_iv_k}\in A$. The reader should be made aware at this point that transitivity will play a key role in what follows in this section.

Definition 1. Let $\mathrm{k_t}(G)$, the *transitive competition number*, be the smallest integer k such that $G\cup I_k$ is a competition graph of a transitive acyclic digraph, where I_k is a set of k isolated vertices added to G.

Define $\mathrm{dk_t}(G)$, the *transitive double competition number*, to be the smallest integer k such that $G\cup I_k$ is a CCE graph of a transitive acyclic digraph, where I_k is a set of k isolated vertices added to G.

In this section, graphs with $\mathrm{k_t}(G)=k$ $(k\geq 0)$ are characterized and examples are given of graphs G with $\mathrm{dk_t}(G)>2$. First, recall some definitions.

Let X be a nonempty set with a partial order $<$ defined on it. Define the following graphs associated with the poset $(X, <)$. The *upper bound graph* (UB-graph) is the graph $U=(X, E(U))$ where $xy\in E(U)$ if and only if $x\neq y$ and there exists an $m\in X$ such that $x, y\leq m$. We say a graph G is a UB-graph if there exists a poset whose upper bound graph is isomorphic to G. The *strict upper bound graph* corresponding to the poset $(X, <)$ has vertex set X and an edge between $x\neq y$ in X if and only if there exists $m \in X$ such that $x, y < m$. The interested reader may want to refer to the characterization of UB-graphs by McMorris and Zaslavsky [6].

The *double bound graph* (DB-graph) of a poset $(X, <)$ is the graph $D = (X, E(D))$ where $xy \in E(D)$ if and only if $x \neq y$ and there exist $m, n \in X$ such that $n \leq x, y \leq m$. We say that a graph G is a DB-graph if there exists a poset whose double bound graph is isomorphic to G. The *strict double bound graph* corresponding to the poset $(X, <)$ has vertex set X and an edge between $x \neq y$ in X if and only if there exist $m, n \in X$ such that $n < x, y < m$. DB-graphs are characterizaed in [3].

As observed by Roberts [8, p. 103], the strict upper bound graph of a poset $(X, <)$ is the competition graph of the transitive, acyclic digraph corresponding to the partial order. ($x < y$ in $(X, <)$ corresponds to $\overrightarrow{xy}$ in the digraph). Similarly, the strict doulbe bound graph of a poset $(X, <)$ is the competition-common enemy graph of the transitive, acyclic digraph corresponding to the partial order. With these observations, the following two facts are evident.

Fact 1. *A graph G is a strict UB-graph if and only if* $\mathrm{k_t}(G) = 0$.

Fact 2. *A graph G is a strict DB-graph if and only if* $\mathrm{k_t}(G) = 0$.

Strict UB-graphs have been characterized by McMorris and Zaslavsky [6]. The astute reader will note that the C_i's in their theorem are sets of vertices, whereas we would take C_i to be the induced subgraph on this same set of vertices. We will use their notation throughout this section. Their characterization is given below in Theorem 5.

Theorem 5. *The graph $G = (V, E)$ is a strict UB-graph if and only if there exists a family $\mathscr{C} = \{C_1, C_2, \dots, C_m\}$ of cliques that edge covers G and $V = C_1 \cup C_2 \cup \cdots \cup C_m \cup \bar{K}_n$ for some $n \geq m$, where $\bar{K}_n$ has no vertices in common with any C_i.*

Define $\theta_e(G)$ to be the smallest number among the cardinalities of all sets which are edge clique covers of G. It will be shown that if $\theta_e(G) = m$, then G must have at least m isolated vertices to be a strict UB-graph.

Theorem 6. *Let $G = (V, E)$ be a graph, and $\mathscr{C} = \{C_1, \dots, C_m)$ be an edge clique cover of G where $\theta_e(G) = m$, and let q be the number of isolated vertices of G. Then* $\mathrm{k_t}(G) = 0$ *if and only if $q \geq m$ and* $\mathrm{k_t}(G) = k$ $(k \geq 0)$ *if and only if $m - q = k$.*

Proof. Suppose $\mathrm{k_t}(G) = 0$. Fact 1 implies that G is a strict UB-graph and it follows from Theorem 5 that $q \geq m$. If $\mathrm{k_t}(G) = k$ $(k \geq 0)$, then $G \cup I_k$ is a strict UB-graph. It then follows from Theorem 5 that $q + k \geq m$ or $m - q \leq k$. But by definition of $\mathrm{k_t}(G)$, k is the smallest integer such that $G \cup I_k$ is a competition graph of a transitive, acyclic digraph. Thus, $m - q = k$.

The converse follows immediately from the fact that $\theta_e(G)=m$. □

In [6], strict DB-graphs were mentioned but not characterized. A characterization for strict DB-graphs might lead to a characterization of graphs for which $\mathrm{dk}_t(G)=k$, $k\geq 0$.

Theorem 7. *Let $G=(V,E)$ be a graph. If there exists a family $\mathscr{C}=\{C_1,\ldots,C_m\}$ of cliques that edge covers G and $V=C_1\cup\cdots\cup C_m\cup\bar{K}_m$, where $n\geq m+1$ and $\bar{K}_n$ has no vertices in common with any C_i, then G is a strict DB-graph.*

Proof. Assign exactly one vertex $a_i\in\bar{K}_n$ *to each clique* $C_i\in\mathscr{C}$ and define the poset $(V,<)$ as follows.

For each a_i, define $v<a_i$ for all $v\in C_i$. There is at least one vertex $a\in\bar{K}_n$ remaining. For each i, set $a<v$, for all $v\in C_i$ $(i=1,2,\ldots,m)$. Clearly, G is the strict DB-graph of $(V,<)$. □

Example 3 will help motivate what follows in Theorem 8.

Example 3. Let $G=K_{1,6}\cup\bar{K}_n$ where $V=\{v_0,v_1,v_2,\ldots,v_6,a_1,a_2,\ldots,a_n\}$, $\bar{K}_n=\{a_i\colon 1\leq i\leq n\}$, and $E=\{v_0v_j\colon 1\leq j\leq 6\}$. Then clearly $\mathscr{C}=\{C_1,C_2,\ldots,C_6\}$, $C_j=\{v_0,v_j\}$, $1\leq j\leq 6$, is an edge clique cover of G and $\theta_e(G)=6$. Consider two cases.

Case 1: n is odd. Then G is a strict DB-graph if $n^2-1\geq 24=4\cdot 6$. An appropriate poset is given in Fig. 4(a) below.

Case 2: n is even. Then G is a strict DB-graph if $n^2\geq 24=4\cdot 6$. An appropriate poset is given in Fig. 4(b). The reader should note that an appropriate poset cannot be found for any even integer n less than 6.

Let G be a strict DB-graph and $\mathscr{C}=\{C_1,\ldots,C_m\}$ be an edge clique cover for G such that $\theta_e(G)=m$. Let $(P,<)$ be any poset realizing G. Throughout the remainder of this section let M be the set of nonisolated maximal elements in $(P,<)$

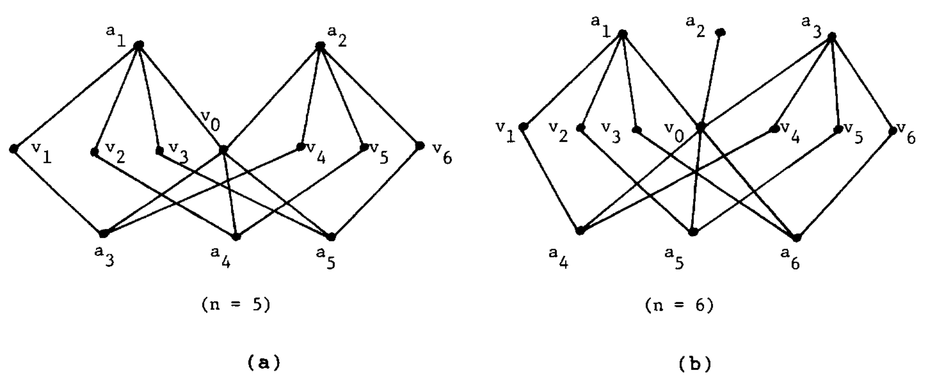

Fig. 4.

and N be the set of nonisolated minimal elements. Then it must be the case that $|M|\cdot|N|\geq m$.

Theorem 8. *If $G=(V,E)$ is a strict DB-graph, then there exist a family $\mathscr{C}=\{C_1,\dots,C_m\}$ of cliques that edge covers G and $V=C_1\cup\cdots\cup C_m\cup\bar{K}_n$, where $\bar{K}_n$ has no vertices in common with any C_i. Furthermore,*

(i) $n^2-1\geq 4m$, *if n is odd,*

(ii) $n^2\geq 4m$, *if n is even.*

Proof. Recall that given an integer n, if x and y are integers whose sum is n and whose product is a maximum, then $x=\lfloor n/2\rfloor$ and $y=\lceil n/2\rceil$.

Let G be a strict DB-graph and $(V,<)$ be a poset with strict DB-graph isomorphic to G. Let M be the set of nonisolated maximal elements of V, and let N be the set of nonisolated minimal elements of V. For convenience, let I be the set of isolated vertices of $(V,<)$. For each $x\in M$, $y\in N$ with $y<x$, define $C(x,y)=\{v\in V: y<v<x\}$. Some $C(x,y)$'s may be empty or singleton sets. However, the collection of these $C(x,y)$'s edge cover G. Let $\mathscr{C}$ be the collection of those $C(x,y)$ which are maximal complete subgraphs in G. Since no element of M or N is contained in any $C(x,y)$, $M\cup N$ is a set of isolated vertices of G. It follows that $\bar{K}_n=M\cup N\cup I$. Let $x=|M|$, $y=|N|$. Then $x+y\leq n$ and $m\leq x\cdot y\leq\lfloor n/2\rfloor\cdot\lceil n/2\rceil$ which equals $n^2-1/4$ if n is odd and $n^2/4$ if n is even. $\square$

Theorems 7 and 8 give upper and lower bounds, respectively, for $\mathrm{dk_t}(G)$. From these it can be shown that for any n, a graph G can be found with $\mathrm{dk_t}(G)\geq n$ as follows. By Theorem 8, $m\leq n^2/4$ where $m=\theta_e(G)$. Let $G=C_m$, then $\theta_e(G)=m$ and $\mathrm{dk_t}(G)\geq n$, where $n\geq\sqrt{4m+1}$ (if n is odd), $\geq\sqrt{4m}=2\sqrt{m}$ (if n is even). Thus, $\mathrm{dk_t}(C_m)\geq 2\sqrt{m}=n$. If G' is the graph in Example 3 without the isolated vertices, then $\mathrm{dk_t}(G')=5$ or 6 (Fig. 4(a) or 4(b), respectively). These are cases where the lower bound n is achieved.

Example 4 gives a graph where the upper bound is assumed (that is, where $\mathrm{dk_t}(G)=m+1$).

Example 4. Let G be the graph in Fig. 5(a). An edge clique cover for G is

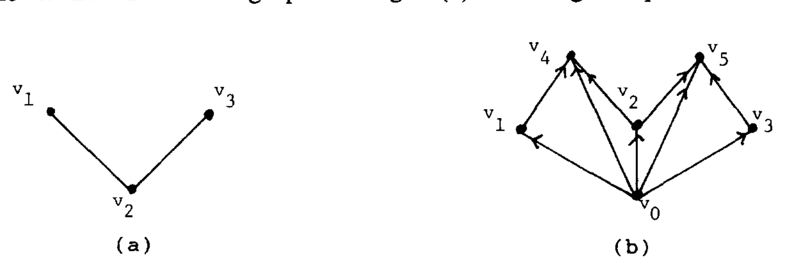

Fig. 5.

$\mathscr{C}=\{C_1,C_2\}$ where $C_1=\{v_1,v_2\}$ and $C_2=\{v_2,v_3\}$. Using Theorem 7, if we add three isolated vertices to G, $G\cup I_3$ is surely a strict DB-graph. Theorem 8 implies that if n is the number of isolated vertices added to G, then for G to be a strict DB-graph, the smallest n could possible be is 3. It follows that $\mathrm{dk_t}(G)=3$ and a transitive, acyclic digraph with CCE graph $G\cup I_3$ is shown in Fig. 5(b).

3. CCE graphs of various digraphs

In this section, graphs G for which $\mathrm{dk}(G)=0$ are characterized, that is, graphs which arise as competition-common enemy graphs of acyclic digraphs. Graphs which are CCE graphs of arbitrary digraphs without loops and digraphs in general (loops allowed) are also characterized.

The following theorem is analogous to a result by Dutton and Brigham [4].

Theorem 9. *$G=(V,E)$, $|V|=n$, is the CCE graph of an acyclic digraph if and only if G has an edge cover by complete subgraphs $\mathscr{C}=\{C_{ij}$: where $1\le i<j\le n\}$ and a labelling of the vertices $v_1,v_2,\dots,v_n$ such that the following hold:*

(i) *$v_k\in C_{ij}$ implies $i<k<j$, and*

(ii) *if $|I_i\cap J_j|>1$, then $I_i\cap J_j=C_{ij}$ where*

$$I_i=\bigcup_{p>i} C_{ip}\cup\{v_b\colon v_i\in C_{ab}, a<i<b\},$$

$$J_j=\bigcup_{q<j} C_{qj}\cup\{v_c\colon v_j\in C_{cd}, c<j<d\}.$$

Proof. Assume G is the CCE graph of an acyclic digraph D. Then since D is acyclic, the vertices can be labelled $v_1,v_2,\dots,v_n$ such that $\overrightarrow{v_iv_j}\in E(D)$ implies $i<j$. Define $C_{ij}=\{v_k\colon \overrightarrow{v_iv_k},\overrightarrow{v_kv_j}\in E(D)\}$. Let $\mathscr{C}$ be the collection of nonempty, nonsingleton C_{ij}'s. Clearly $\mathscr{C}$ is an edge cover for G and the C_{ij} are complete subgraphs of G. Moreover, condition (i) is satisfied because of the choice of the labelling of vertices in D. It remains to show condition (ii) holds.

Fix i and j and let I_i and J_j be sets as defined in the theorem. Since $C_{ij}\subseteq I_i$ and $C_{ij}\subseteq J_j$, clearly $C_{ij}\subseteq I_i\cap J_j$. Now assume $|I_i\cap J_j|>1$ and let $v_k\in I_i\cap J_j$. There are four cases for v_k arising from the definitions of I_i and J_j, namely,

(1) $v_k\in(\bigcup_{p>i}C_{ip})\cap(\bigcup_{q<j}C_{qj})$,

(2) $v_k\in(\bigcup_{p>i}C_{ip})\cap\{v_c\colon v_j\in C_{cd}, c<j<d\}$,

(3) $v_k\in\{v_b\colon v_i\in C_{ab}, a<i<b\}\cap(\bigcup_{q<j}C_{qj}\}$, and

(4) $v_k\in\{v_b\colon v_i\in C_{ab}, a<i<b\}\cap\{v_c\colon v_j\in C_{cd}, c<j<d\}$.

We prove that $I_i\cap J_j\subseteq C_{ij}$ for case (1). The other cases are similar.

Let $v_k\in C_{ip}$, some $p>i$, and $v_k\in C_{qj}$, some $q<j$. By the definition of C_{ij}, $v_k\in C_{ip}$ implies $\overrightarrow{v_iv_k},\overrightarrow{v_kv_p}\in E(D)$. Likewise, $v_k\in C_{qj}$ implies $\overrightarrow{v_qv_k},\overrightarrow{v_kv_j}\in E(D)$. Thus $\overrightarrow{v_iv_k},\overrightarrow{v_kv_j}\in E(D)$ and $v_k\in C_{ij}$. Since $v_k\in I_i\cap J_j$ is arbitrary, $I_i\cap J_j\subseteq C_{ij}$.

Conversely, let G have an edge cover $\mathscr{C}=\{C_{ij}$: where $1\le i<j\le n\}$ by complete

subgraphs of G and a labelling of the vertices $v_1, v_2, \ldots, v_n$ such that (i) and (ii) hold. (Observe that G must have at least two isolated vertices, namely, v_i and v_n by condition (i).) Define D as follows: $V(D) = V$ and $\overrightarrow{v_i v_k}, \overrightarrow{v_k v_j} \in E(D)$, for all $v_k \in C_{ij}$. Condition (i) guarantees that D is acyclic. Since endpoints of every edge in G are contained in some C_{ij}, there is a common prey v_j and a common enemy v_i for vertices of an edge.

Now suppose $\overrightarrow{v_i v_k}, \overrightarrow{v_i v_r}, \overrightarrow{v_k v_j}, \overrightarrow{v_r v_j} \in E(D)$. We show $v_k v_r \in E(G)$. Since $\overrightarrow{v_i v_k}, \overrightarrow{v_i v_r} \in E(D)$, it follows $\{v_k, v_r\} \subseteq I_i$. Likewise, $\overrightarrow{v_k v_j}, \overrightarrow{v_r v_j} \in E(D)$ implies $\{v_k, v_r\} \subseteq J_j$. Thus, $\{v_k v_r\} \subseteq I_i \cap J_j = C_{ij}$ and this implies $v_k v_r \in E(G)$. Hence, G is the CCE graph of D. □

Part of the difficulty in finding a graph G with $\mathrm{dk}(G) > 2$ is that if G is any graph on n vertices with two isolated vertices labelled v_1 and v_n, we can always find an edge cover of complete subgraphs satisfying condition (i) as follows. If $\overrightarrow{v_k v_m}$ is an arc with $k < m$, let $C_{k-1,m+1} = \{v_k, v_m\}$, the complete subgraph consisting of just the arc $\overrightarrow{v_k v_m}$. Then clearly the set of such C_{ij} edge covers G and satisfies condition (i). However, this cover does not necessarily satisfy condition (ii), and frequently does not.

In Theorem 10, Corollary 5 and Corollary 6 below, I_i and J_j are defined as

$$I_i = \bigcup_p C_{ip} \cup \{v_b : v_i \in C_{ab}\} \quad \text{and} \quad J_j = \bigcup_q C_{qj} \cup \{v_c : v_j \in C_{cd}\}.$$

Theorem 10 is analogous to a result by Roberts and Steif [9] and is very similar to Theorem 9 with condition (i) modified.

Theorem 10. *$G = (V, E)$, $|V| = n$, is the CCE graph of a digraph without loops if and only if G has an edge cover $\mathscr{C} = \{C_{ij} : i, j \in \{1, 2, \ldots, n\}\}$ by complete subgraphs and a labelling of the vertices $v_1, v_2, \ldots, v_n$ such that the following hold:*

(i) *$v_i, v_j \notin C_{ij}$, and*

(ii) *if $|I_i \cap J_j| > 1$, then $I_i \cap J_j = C_{ij}$.*

Proof. Assume G is the CCE graph of a digraph D without loops. Define for $i \neq j$, $C_{ij} = \{v_k : \overrightarrow{v_i v_k}, \overrightarrow{v_k v_j} \in E(D)\}$. If $\mathscr{C} = \{C_{ij} : |C_{ij}| > 1\}$, then clearly $\mathscr{C}$ edge covers G and the $C_{ij} \in \mathscr{C}$ are complete subgraphs of G. Since D has no loops, $v_i, v_j \notin C_{ij}$ and (i) holds. Condition (ii) is checked similar to that in Theorem 9 above.

The converse follows analogously to the proof of Theorem 9. □

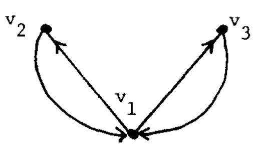

Fig. 6.

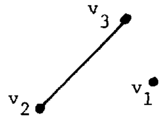

Fig. 7.

Observe that condition (i) does not exclude the possibility of C_{ii} being a complete subgraph for consider the digraph D in Fig. 6. D has no loops and has CCE graph G shown in Fig. 7. Taking $\mathscr{C}=\{C_{11}\}$, where $C_{11}=\{v_2,v_3\}$, the conditions of the Theorem are satisfied.

It should be pointed out to the reader that different digraphs can be constructed (as in the proof of Theorem 10) with the same CCE graph if different labels are chosen for the complete subgraphs in the edge cover, so long as conditions (i) and (ii) of Theorem 10 are satisfied.

A characterization for CCE graphs of arbitrary digraphs, loops allowed, follows immediately from Theorem 9 and Theorem 10.

Corollary 5. *$G=(V,E)$, $|V|=n$, is the CCE graph of an arbitrary digraph (loops allowed) if and only if G has an edge cover by complete subgraphs, $\mathscr{C}=\{C_{ij}: i,j\in\{1,\dots,n\}\}$ and a labelling of the vertices $v_1,v_2,\dots,v_n$ such that for any $i,j\in\{1,2,\dots,n\}$,*

if $|I_i\cap J_j|>1$, then $I_i\cap J_j=C_{ij}$.

A reflexive digraph, a digraph with loops on all the vertices, is such that its CCE graph includes all edges of the digraph (considered undirected in the CCE graph), with possible additional edges. Adding one condition to Corollary 5, a characterization of CCE graphs of reflexive digraphs follows.

Corollary 6. *$G=(V,E)$, $|V|=n$, is the CCE graph of a reflexive digraph if and only if there exists a set of complete subgraphs $\mathscr{C}=\{C_{ij}: i,j\in\{1,2,\dots,n\}\}$ that edge covers G and a labelling of the vertices $v_1,v_2,\dots,v_n$ such that the following hold:*

(i) *For all $v_i\in V$, $v_i\in(\bigcup_b C_{ib})\cup(\bigcup_a C_{ai})$,*

(ii) *If $|I_i\cap J_j|>1$, then $I_i\cap J_j=C_{ij}$.*

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